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Anomalous dimension and local charges

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Abstract: AdS space is the universal covering of a hyperboloid. We consider the action of the deck transformations on a classical string worldsheet in $AdS_5 \times S^5$. We argue that these transformations are generated by an infinite linear combination of the local conserved charges. We conjecture that a similar relation holds for the corresponding operators on the field theory side. This would be a generalization of the recent field theory results showing that the one loop anomalous dimension is proportional to the Casimir operator in the representation of the Yangian algebra.

Keywords: Integrable Equations in Physics, Space-Time Symmetries, Differential and Algebraic Geometry, AdS-CFT Correspondence.

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Contents

1. Introduction

The development of the AdS/CFT correspondence was originally obstructed by our poor understanding of the string worldsheet theory in the background with the nonzero Ramond-Ramond field strength. But recently it was realized that the string worldsheet theory has a beautiful mathematical structure related to the integrability. Classical integrability of the string worldsheet theory first discussed in the context of the AdS/CFT correspondence in [1, 2] should play an important role in the quantization of the worldsheet theory. If we learn the proper use of these integrable structures, this could perhaps even give us a fresh perspective on the string perturbation theory.

Integrable structures were also found in the dual $\mathcal{N} = 4$ supersymmetric planar Yang-Mills theory. Even before the discovery of the AdS/CFT correspondence, integrability was found in the high energy sector of QCD in [3, 4]; see [5] for a recent discussion of the past results and their relation to the string theory. In [6−9] the one-loop anomalous dimension of single trace operators in the $\mathcal{N} = 4$ theory was computed; it was found that the anomalous dimension is given by the hamiltonian of an integrable spin chain. Moreover, some evidence of the integrability at the level of two loops was found; see [10] for the review and the list of references.

Infinite dimensional symmetry algebras known as Yangians play an important role in the theory of integrable systems [11]. They are not Lie algebras, but rather associative algebras. One of the most important properties of the Yangians is the existence of the comultiplication which allows to introduce a tensor product of representations. The comultiplication makes the Yangian a very natural algebra to consider in the theory of spin chains. Indeed, the space of states of the spin chain is the tensor products of the "elementary" spaces over its sites. Usually Yangian acts in some elementary way on each site, and then we use the comultiplication to extend its action to the whole chain. The action of the Yangian is encoded in the *transfer matrix* $T(z)$, which depends on the *spectral parameter* z. The matrix elements of the transfer matrix are not the c-numbers, but the linear operators acting in the space of states of the spin chain. In other words, the Yangian algebra is generated by the matrix elements of the transfer matrix $T(z)$; the coefficients of the Taylor expansion of the matrix elements in powers of z generate the Yangian. The defining relations of the Yangian are encoded in the formula $R(z_1 - z_2)T(z_1)T(z_2) = T(z_2)T(z_1)R(z_1 - z_2)$ where $R(z)$ is some matrix whose elements are complex numbers. The form of this matrix is restricted by the conditions of the "nontriviality" of the algebra, known as the Yang-Baxter equations.

In the context of the AdS/CFT correspondence the Yangian symmetry was first discussed on the string theory side in [2]. The study of the Yangian symmetry on the field theory side was initiated in [12, 13]. It was conjectured that the Yangian algebra acts on the single trace states in the weakly coupled gauge theory. (Strictly speaking, one has to consider operators constructed of infinitely many elementary fields, in order to avoid some "edge effects". The action of the Yangian requires a linear ordering of the sites of the chain, not just a cyclic ordering.) The explicit expression for the action in the zero coupling limit was conjectured, motivated by the theory of spin chains. In [14] the one-loop anomalous dimension was expressed in terms of the local conserved charges. Local conserved charges can be obtained as the coefficients of the Laurent series of $\log \mathrm{tr} T(z)$ at the point $z = z_0$ where $T(z)$ has a singularity[1]. They could be thought of as Casimir operators of the Yangian. Even though the Yangian itself is not well defined on the single trace states of a finite engineering dimension, the Casimirs are well defined.

It would be interesting to extend the relation between the anomalous dimension and the Casimirs to higher loops. The anomalous dimension is usually defined as the deformation of the particular generator of the conformal algebra — the dilatation operator. But in fact the anomalous dimension parametrizes the deformation of the *representation* of the conformal algebra, rather than the deformation of a particular generator. It is more natural to define the anomalous dimension through the action of the center of the conformal group. This definition is manifestly conformally invariant. Notice that the coherent single-trace states corresponding to the classical strings [15] usually do not have a definite engineering dimension [16], therefore the "standard" definition will not work for them.

In section 2 we will explain how the anomalous dimension can be defined through the action of the center of the conformal group. In section 3 we will show that this definition is very natural from the string theory point of view and implies that the anomalous dimension is an infinite sum of the local conserved charges.

2. Anomalous dimension as a deck transformation

2.1 Field theory side

The bosonic part of the supergroup $\mathrm{PSU}(2,2|4)$ is not simply connected; the superconformal group of the conformal field theory is actually a covering group which we will denote $\widetilde{\mathrm{PSU}}(2,2|4)$. The bosonic part of $\widetilde{\mathrm{PSU}}(2,2|4)$ is $[\widetilde{SU}(2,2)\times \mathrm{SU}(4)]/\mathbf{Z}_2$. We denoted $\widetilde{SU}(2,2)$ the universal covering of $\mathrm{SU}(2,2)$, and $\mathbf{Z}_2$ is generated by $a \times (-\mathbf{1}) \in \widetilde{SU}(2,2) \times \mathrm{SU}(4)$ where a is the rotation of the sphere S^3 by the angle 2π around one of its axes. (Notice that the bosonic part of the superconformal group is not simply connected; it has $\pi_1 \simeq \mathbf{Z}_2$.)

[1]I want to thank G. Arutyunov for explaining this to me.

Let c denote the generator of the center. The action of c can be understood in the following way. Consider the conformal field theory on $\mathbf{R}\times S^3$ where $\mathbf{R}$ is the time and the radius of S^3 is 1. Let t denote the time and $\vec{n}$ denote the unit vector parametrizing S^3. Then c acts as the combination of the conformal transformation:

$$c:\ (t,\vec{n})\to(t+\pi,-\vec{n}) \tag{2.1}$$

with the R-symmetry $i\mathbf{1}\in \mathrm{SU}(4)$. This transformation commutes with the generators of $so(2,4)$ and therefore it is in the center of the conformal group. It also commutes with the fermionic generators of the supersymmetry, therefore it is in the center of $\widetilde{\mathrm{PSU}}(2,2|4)$.

If we represent the elements of the group $\mathrm{U}(2,2|4)$ by the $(4|4)\times(4|4)$ matrices, then c will correspond to the central element:

$$c=\begin{pmatrix} i\mathbf{1}_{4\times4} & \mathbf{0}_{4\times4}\\ \mathbf{0}_{4\times4} & i\mathbf{1}_{4\times4}\end{pmatrix} \tag{2.2}$$

But to describe the superconformal group, we have to work on the covering space of the space of matrices. Therefore, the matrix (2.2) should be supplemented with the path connecting it to the unit matrix, two choices of the path considered equivalent if they can be smoothly deformed to each other. The central element c corresponds to the following path:

$$C(t)=\begin{pmatrix} \mathrm{diag}\left(e^{-\frac{3\pi i}{2}t},e^{\frac{\pi i}{2}t},e^{\frac{\pi i}{2}t},e^{\frac{\pi i}{2}t}\right) & \mathbf{0}_{4\times4}\\ \mathbf{0}_{4\times4} & \mathrm{diag}\left(e^{-\frac{3\pi i}{2}t},e^{\frac{\pi i}{2}t},e^{\frac{\pi i}{2}t},e^{\frac{\pi i}{2}t}\right)\end{pmatrix} \tag{2.3}$$

parametrized by $t\in[0,1]$, $C(0)=\mathbf{1}$, $C(1)=c$.

Notice that c^2 acts as the shift of the time $t\to t+2\pi$ combined with $(-1)^F$. In the free field theory $c=1$ but in the interacting theory c acts nontrivially. We will define the anomalous dimension through the action of c^2:

$$c^2=e^{2\pi i\Delta E}\,. \tag{2.4}$$

In perturbation theory ΔE is expanded in powers of λ. We will call ΔE the anomalous dimension. We could also define ΔE through the action of c, but the corresponding definition on the string theory side would be somewhat less transparent.

This definition is equivalent to the definition throught the deformation of the dilatation generator for the operators which have a definite engineering dimension. Indeed, the correction due to the interaction to the eigenvalue of the dilatation operator on the local operator $\mathcal{O}$ in the euclidean theory is equal to the energy shift for the corresponding state ψ in the theory on $\mathbf{R}\times S^3$. The energy is the eigenvalue of a generator of the superconformal group which we will denote H. Geometrically H generates shifts of the global time $\mathbf{R}$. Suppose that ψ is an eigenstate of the hamiltonian:

$$H\psi=E\psi\,. \tag{2.5}$$

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Let E_0 be the energy of this state in the free theory (the engineering dimension of the corresponding operator). Then in the interacting theory $E = E_0 + \Delta E$ where ΔE is the energy shift. In the Yang-Mills perturbation theory $\Delta E \simeq \lambda$. We have

$$e^{2\pi iH}\psi = e^{2\pi iE}\psi = e^{2\pi i(E_0+\Delta E)}\psi = (-1)^F e^{2\pi i\Delta E}\psi \tag{2.6}$$

since E_0 is an integer or a half-integer depending on whether the state is bosonic or fermionic. In perturbation theory $\Delta E << 1$ and therefore we can write

$$\Delta E = \frac{1}{2\pi i}\log\left((-1)^F e^{2\pi iH}\right). \tag{2.7}$$

Notice that $e^{2\pi iH}$ generates the shift $t \to t + 2\pi$. The operator $(-1)^F e^{2\pi iH}$ is in the center of the superconformal group (it commutes with all the generators of the superconformal group). This operator represents on the space of states the discrete symmetry which we have denoted c^2. Therefore eq. (2.7) implies eq. (2.4).

2.2 AdS side

The AdS space is the universal covering space of the hyperboloid and c^2 acts as a deck transformation exchanging the sheets. We can visualize the action of this deck transformation on the string phase space in the following way. Let us replace AdS_5 by the hyperboloid $H = AdS_5/\mathbf{Z}$. Let us formally consider the string as living on $(AdS_5/\mathbf{Z}) \times S^5$. Pick a point x on the string worldsheet Σ. Consider a neighborhood of x in $(AdS_5/\mathbf{Z}) \times S^5$ which is simply connected. An example of such a neighborhood is a set of points which are within the distance $R/2$ from x, where R is the radius of AdS_5. Let B denote such a neighborhood. Consider the part of the string worldsheet which is inside B (that is, $B \cap \Sigma$). One can see that $B \cap \Sigma$ consists of several sheets, which can be enumerated. These sheets are two-dimensional, so we can think of them as cards; $B \cap \Sigma$ is then a deck of cards, see figure 1.

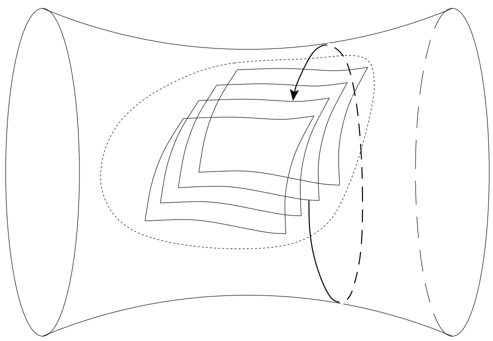

Figure 1: The string worldsheet looks locally like a deck of cards. Going around the noncontractible cycle in the hyperboloid exchanges the sheets. This is the *deck transformation*. It measures the deviation of the string worldsheet from being periodic in the global time.

Let x belong to the sheet number n, then we can draw a path on Σ starting at x, winding once on the noncontractible cycle in $AdS_5/\mathbf{Z}$ and then ending on the sheet number $n+1$. Let Σ_n denote the sheet number n. The deck transformation maps Σ_n to Σ_{n+1}, Σ_{n+1} to Σ_{n+2} and so on. This determines the action of c^2 on the phase space of the classical string.

3. Deck transformations and local charges

Classical strings in AdS space correspond to classical single-trace states on the field theory side. In the classical regime, the length of the operator (the number of the elementary fields under the trace) was conjectured to be conserved. The corresponding conserved charge was constructed in [17] using the hamiltonian perturbation theory around the null-surfaces. We have conjectured in [18] that this conserved charge is an infinite linear combination of the local conserved charges which are known as Pohlmeyer charges [19]:

$$L = \sqrt{\lambda}\,[\mathcal{E}_2 + a_1\mathcal{E}_4 + a_2\mathcal{E}_6 + a_3\mathcal{E}_8 + \ldots]\,. \tag{3.1}$$

The coefficients a_n can in principle be extracted[2] from the known explicit expressions for the charges of the rigid strings (see [20, 21] and the discussion in [18]) but it should be possible to find a general interpretation of this conserved charge in the framework of the Bethe ansatz [22−25]. The characteristic property of L is that the corresponding hamiltonian vector field ξ_L has periodic trajectories:

$$e^{2\pi\xi_L} = \text{identical transformation}\,. \tag{3.2}$$

Therefore L is an action variable.

The dynamics of the classical string in $AdS_5 \times S^5$ essentially splits into the direct product of two systems: the sigma-model with the target space AdS_5 and the sigma-model with the target space S^5. In eq. (3.1) $\mathcal{E}_{2k}$ are the Pohlmeyer charges for the S^5 sigma-model. One can construct in the same way the Pohlmeyer charges for the AdS_5 sigma-model. Let us denote them $\mathcal{F}_{2k}$. Consider the conserved charge K:

$$K = \sqrt{\lambda}\,[\mathcal{F}_2 + a_1\mathcal{F}_4 + a_2\mathcal{F}_6 + a_3\mathcal{F}_8 + \ldots] \tag{3.3}$$

defined with the same coefficients a_k as in (3.1). This charge would also generate the periodic trajectories if we formally considered the string on $(AdS_5/\mathbf{Z}) \times S^5$. We would then have $e^{-2\pi\xi_K}$ acting as the identity map. But for the string on $AdS_5 \times S^5$ it acts as the deck transformation:

$$e^{-2\pi\xi_K} = c^2\,. \tag{3.4}$$

Indeed, since AdS is the universal cover of the hyperboloid, the fact that the canonical transformation $e^{-2\pi\xi_K}$ acts as an identical map on the string on the hyperboloid implies that it acts on the string on AdS_5 as some iteration of the deck transformation. In fact it is the first iteration of the deck transformation. To understand why it is the first iteration (and not, for example, the second iteration c^4) it is enough to consider its action on the null-surface. The projection of the null-surface to the hyperboloid $AdS_5/\mathbf{Z}$ is a continuous collection of equators of the hyperboloid. Let us specify the null-surface by its embedding $x_0(\tau,\sigma)$; for a fixed $\sigma = \sigma_0$ the curve $x_0(\tau,\sigma_0)$ is a light ray. We will denote $x_{0,S}$ the

[2]Note added in the revised version: the coefficients a_n are calculated from the plane wave limit in [29].

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projection of the string worldsheet to the sphere, and $x_{0,A}$ the projection to AdS. We will use the worldsheet coordinates with the property $(\partial_\tau x_{0,A}, \partial_\tau x_{0,A}) = -1$. The vector field ξ_K acts as an infinitesimal shift along the equator:

$$\begin{aligned} \xi_K.\, x_{0,S} &= 0 \\ \xi_K.\, x_{0,A} &= \partial_\tau x_{0,A} \end{aligned} \tag{3.5}$$

Therefore $e^{-\alpha\xi_K}$ acts as a shift by an angle α along the equator. When α grows from 0 to 2π the point $[e^{-\alpha\xi_K}x_0](\tau,\sigma)$ goes all the way around the corresponding equator of AdS. As a point on the hyperboloid, it returns when $\alpha = 2\pi$ back to $x_0(\tau,\sigma)$; but as a point on AdS it goes around the noncontractible cycle and ends up on the other cover of the hyperboloid. This means that $e^{-2\pi\xi_K}$ acts as the first iteration of the deck transformation on the null-surfaces. Since the null-surfaces can be approximated by the fast moving strings, this implies by the continuity that $e^{-2\pi\xi_K}$ acts as c^2 on the string if the string moves fast enough.

Notice that ξ_K commutes with ξ_L. Therefore

$$c^2 = e^{2\pi(\xi_L - \xi_K)}\,. \tag{3.6}$$

The hamiltonian flow of $\mathcal{E}_2 - \mathcal{F}_2$ acts trivially (this combination generates reparametrizations on the string worldsheet; the Virasoro constraints require that $\mathcal{E}_2 = \mathcal{F}_2$). Therefore we can identify

$$\frac{1}{2\pi}\log c^2 = \sqrt{\lambda}\,[a_1(\mathcal{E}_4 - \mathcal{F}_4) + a_2(\mathcal{E}_6 - \mathcal{F}_6) + a_3(\mathcal{E}_8 - \mathcal{F}_8) + \ldots]\,. \tag{3.7}$$

This expression is a perturbative expansion of the anomalous dimension of the fast moving string in the null-surface perturbation theory [26, 27]. The small parameter is the relativistic factor $\sqrt{1-v^2}$, where v is the typical velocity of the string. One can define the local conserved charges in such a way that $\mathcal{E}_{2k}$ and $\mathcal{F}_{2k}$ are of the order $(1-v^2)^{k-3/2}$ and depend on the embedding coordinates $x(\tau,\sigma)$ and their derivatives with respect to τ and σ up to the order k. Eq. (3.1) and the fact that $\mathcal{E}_2 \sim (1-v^2)^{-1/2}$ imply that $\sqrt{1-v^2} \sim \frac{\sqrt{\lambda}}{L}$ and therefore (3.7) is an expansion in powers of $\frac{\lambda}{L^2}$. The first term is of the order $\frac{\lambda}{L}$, the second term is of the order $\frac{\lambda^2}{L^3}$ and so on [28]. The zeroth approximation is the null-surface corresponding to the infinitely long operator which has a zero anomalous dimension.

It is not clear to us if the expansion (3.7) converges and defines the action of the deck transformations beyond the perturbation theory. The local conserved charges do not exhaust all the commuting hamiltonians of the sigma-model, but the other conserved charges are nonlocal. In the perturbation theory we know from [17, 18] that L is a local functional in each order of the perturbation theory. Therefore the local charges should be enough to construct L in the perturbation theory.

On the field theory side $\mathcal{E}_{2k} - \mathcal{F}_{2k}$ should be identified with some combinations of the Casimir operators of the Yangian (more precisely, the purely bosonic parts of the Casimir operators of the super-Yangian of PSU$(2,2|4)$). It would be very interesting to extend the calculation of [14] to higher loops and write the formula analogous to (3.7) on the field theory side.

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